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Control of a network of Euler–Bernoulli beams

D. Mercier, V. Régnier *

*Laboratoire de Mathématiques et ses Applications de Valenciennes, FR CNRS 2956, Institut des Sciences et Techniques de Valenciennes,
 Université de Valenciennes et du Hainaut-Cambrésis, Le Mont Houy, 59313 Valenciennes Cedex 9, France*

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Abstract

The aim is to study the boundary controllability of a system modeling the vibrations of a network of N Euler–Bernoulli beams connected by n vibrating point masses. Using the classical Hilbert Uniqueness Method, the control problem is reduced to the obtention of an observability inequality. The solution is then expressed in terms of Fourier series so that it is also enough to show that the distance between two consecutive large eigenvalues of the spatial operator involved in this evolution problem is superior to a minimal fixed value. This property called spectral gap holds as soon as the roots of a function denoted by f_∞ (and giving the asymptotic behaviour of the eigenvalues) are all simple. For a network of $N = 2$ different beams, this assumption on the multiplicity of the roots of f_∞ (denoted by (A)) is proved to be satisfied and controllability follows. For higher values of N , a numerical approach allows one to prove (A) in many situations and no counterexample has been found but the problem of giving a general proof of controllability remains open.

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1. Introduction

In the last few years various physical models of multi-link flexible structures consisting of finitely many interconnected flexible elements such as strings, beams, plates, shells have been mathematically studied. See [11,12,17,25,27] for instance. The spectral analysis of such structures has some applications to control or stabilization problems [25,26]. For interconnected strings (corresponding to a second-order operator on each string), a lot of results have been obtained: the asymptotic behaviour of the eigenvalues [1,2,10,34], the relationship between the eigenvalues and algebraic theory (cf. [7,8,25,33]), qualitative properties of solutions (see [10] and [36]) and finally studies of the Green function (cf. [22,37,39]).

For interconnected beams (corresponding to a fourth-order operator on each beam), some results on the asymptotic behaviour of the eigenvalues and on the relationship between the eigenvalues and algebraic theory were obtained by Nicaise and Dekoninck in [19,20] and [21] with different kinds of connections using the method developed by von Below in [7] to get the characteristic equation associated to the eigenvalues.

* Corresponding author.

E-mail addresses: denis.mercier@univ-valenciennes.fr (D. Mercier), virginie.regnier@univ-valenciennes.fr (V. Régnier).

The authors used the same method in a recent paper [32] to compute the spectrum for a hybrid system of N flexible beams connected by n vibrating point masses. This type of structure was studied by Castro and Zuazua in many papers (see [13–16,18]) and Castro and Hansen [23]. They have restricted themselves to the case of two beams applying their results on the spectral theory to controllability. They have shown that if the constant of rotational inertia is positive, due to the presence of the mass, the system is well-posed in asymmetric spaces (spaces with different regularity on both sides of the mass) and consequently, the space of controllable data is also asymmetric. For a vanishing constant of rotational inertia the system is not well-posed in asymmetric spaces and the presence of the point mass does not affect the controllability of the system.

Note that S.W. Taylor proved similar results at the same time in [40] using different techniques based on the method presented in [30] for exact controllability.

The authors have also been working on transmission problems on networks for a few years: Mercier studied in [31] transmission problems for elliptic systems in the sense of Agmon–Douglis–Nirenberg on polygonal networks with general boundary and interface conditions.

In [5], Régnier and Ali Mehmeti studied the spectral solution of a one-dimensional Klein–Gordon transmission problem corresponding to a particle submitted to a potential step and interpreted the phase gap between the original and reflected term in the tunnel effect case as a delay in the reflection of the particle. At the same time in [38], Régnier extended this technique to a two-dimensional problem which had been first studied from a spectral point of view by Croc and Dermenjian.

Let us finally quote the paper by Nicaise and Valein [35] on stabilization of the one-dimensional wave equation with a delay term in the feedbacks. They use the same method as we did in our last paper [32] (technique developed by von Below in [7]) to get the characteristic equation associated to the eigenvalues and apply this spectral analysis to stabilization.

In this paper we will still investigate the same problem as in [32] but with different methods which are more adapted to the study of controllability. The way we computed the spectrum in [32] is too complicated to get results about boundary controllability which is our point here.

Let us recall the situation. On a finite network made of N edges with index $j = 1, \dots, N$ and n vertices denoted by E_i with index $i = 1, \dots, n$ (point masses with mass M_i if E_i is an interior vertex of the network), we consider the control problem (PC):

$$\left\{ \begin{array}{l} u_{j,tt}(x,t) + a_j u_{jx_j^{(4)}}(x,t) = 0, \quad \forall j \in \{1, \dots, N\}, \\ u_{j,tt}(E_i, t) - \frac{1}{M_i} \left(\sum_{j \in N_i} a_j \frac{\partial^3 u_j}{\partial v_j^3}(E_i) \right) = 0, \quad \forall j \in \{1, \dots, N\}, \quad \forall i \in I_{\text{int}}, \\ u_j(E_i) = z_i, \quad \forall i \in I_{\text{int}}, \quad \forall j \in N_i, \\ \sum_{j \in N_i} \frac{\partial u_j}{\partial v_j}(E_i) = 0, \quad \forall i \in I_{\text{int}}, \\ a_l \frac{\partial^2 u_l}{\partial v_l^2}(E_i) = a_j \frac{\partial^2 u_j}{\partial v_j^2}(E_i), \quad \forall i \in I_{\text{int}}, \quad \forall (l, j) \in N_i^2, \\ u_j(E_i) = 0, \quad \forall i \in I_{\text{ext}}, \quad \forall j \in N_i, \\ \frac{\partial^2 u_j}{\partial v_j^2}(E_i) = 0, \quad \forall i \in I_{\text{ext}} - \{i_0\}, \quad \forall j \in N_i, \quad \text{and} \quad \frac{\partial^2 u_j}{\partial v_j^2}(E_{i_0}) = q, \quad \forall j \in N_{i_0}. \end{array} \right.$$

The scalar functions $u_j(x, t)$ and $z_i(t)$ contain the information on the vertical displacements of the beams ($1 \leq j \leq N$) and of the point masses ($1 \leq i \leq n$). These displacements are described by the first two equations where the a_j 's are mechanical constants, I_{int} (respectively I_{ext}) is the set of indices corresponding to the interior (respectively exterior) vertices of the network, N_i is the set of edges adjacent to the vertex E_i .

The third, fourth and fifth equations are transmission conditions. The sixth and seventh ones are boundary conditions.

Note that the control function $q = q(t)$ acts on the system through the exterior node E_{i_0} on the quantity $\frac{\partial^2 u_j}{\partial v_j^2}$.

The problem of exact controllability can be formulated as follows: *for any time $T > 0$, find the class \mathcal{H} of initial conditions for which there exists a control function q in $L^2(0, T)$ such that the solution of Problem (PC) is at rest at time $t = T$ i.e.*

$$\begin{cases} u_j(x, T) = 0, & \forall x \in k_j, j \in \{1, \dots, N\}, \\ u_{j,t}(x, T) = 0, & \forall x \in k_j, j \in \{1, \dots, N\} \end{cases}$$

and

$$\begin{cases} z_i(T) = 0, & \forall i \in I_{\text{int}}, \\ z_{i,t}(T) = 0, & \forall i \in I_{\text{int}} \end{cases}$$

(k_j denotes the j th edge of the network).

Before starting to study the core of the problem, we recall in Section 2 the terminology of networks as they can be found in early contributions of Lumer and Gramsch as well as in papers by Ali Mehmeti [3,4], von Below [7] and Nicaise [6,33] in the eighties. We also recall some properties of the spatial operator \mathcal{A} involved in the considered evolution problem (cf. Lemma 1).

In Section 3, the solution of the uncontrolled problem ((PC) with $q = 0$) is expressed in terms of Fourier series and its existence, uniqueness and regularity is established in spaces which involve the domains of the powers of the operator \mathcal{A} . These spaces are also characterized following Castro and Zuazua [16]. Descriptions in terms of Sobolev spaces are given (cf. Proposition 4).

Section 4 is devoted to proving that it is enough to get an observability inequality for controllability to hold. This classical result is an application of the Hilbert Uniqueness Method to our situation (see Lemma 8). A result due to Haraux (cf. [24]) is recalled which states that it is sufficient for the spectrum of the operator to have a particular asymptotic behaviour (called spectral gap) to get the required observability inequality and so, controllability.

The study of the asymptotic behaviour of the eigenvalues of the operator \mathcal{A} is thus envisaged in Section 5. This behaviour is given by that of the roots of a function called f_∞ , which is proved to be a trigonometric polynomial. Its general expression is not easy to obtain. For a network of $N = 2$ different branches, we are able to prove that the roots of f_∞ are simple and satisfy the property called spectral gap. Hence the exact controllability for a class of initial conditions called $\mathcal{H}_{1/4}$ involving the spaces described in Proposition 4 of Section 3 (cf. Theorem 17).

As for higher values of N , the situation is far more complicated. It is still enough to establish that the roots of f_∞ are simple (assumption denoted by (A)). An approach using symbolic computation allows one to get the expression of f_∞ for many examples and prove (A), hence the exact controllability. Since no counterexamples have been found, we conjecture that exact controllability always holds but this remains to be proved.

2. Preliminaries

2.1. Terminology of networks

Let us first introduce some notation and definitions which will be used throughout the rest of the paper, in particular some which are linked to the notion of C^ν -networks, $\nu \in \mathbb{N}$ (as introduced in [9] and recalled in [20]):

All graphs considered here are non-empty, finite and simple. Let Γ be a connected topological graph embedded in \mathbb{R}^m , $m \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, with n_0 vertices and N edges ($(n_0, N) \in (\mathbb{N}^*)^2$). (Note that in concrete applications, $1 \leq m \leq 3$ even if the theory developed by Lumer has been established for $m \geq 1$.)

We split the set E of vertices as follows: $E = E_{\text{int}} \cup E_{\text{ext}}$ where $E_{\text{int}} = \{E_i: 1 \leq i \leq n\}$ is the set of interior vertices and $E_{\text{ext}} = \{E_i: n+1 \leq i \leq n_0\}$ the set of exterior vertices of Γ .

Let $K = \{k_j: 1 \leq j \leq N\}$ be the set of the edges of Γ . Each edge k_j is a Jordan curve in \mathbb{R}^m and is assumed to be parametrized by its arc length x_j such that the parametrization

$$\pi_j: [0, l_j] \rightarrow k_j: x_j \mapsto \pi_j(x_j)$$

is ν -times differentiable i.e. $\pi_j \in C^\nu([0, l_j], \mathbb{R}^m)$ for all $1 \leq j \leq N$. The length of the edge k_j is l_j .

The C^ν -network G associated with Γ is then defined as the union

$$G = \bigcup_{j=1}^N k_j.$$

The valency of each vertex E_i is the number of edges containing the vertex E_i and is denoted by $\gamma(E_i)$. Clearly it holds $E_{\text{int}} = \{E_i: \gamma(E_i) > 1\}$ and $\partial E = E_{\text{ext}} = \{E_i \in E: \gamma(E_i) = 1\}$. For shortness, we later on denote by I_{int} (respectively I_{ext}) the set of indices corresponding to the interior (respectively exterior) vertices i.e. $I_{\text{int}} = \{i: i \in \{1, \dots, n\}\}$ and $I_{\text{ext}} = \{i: i \in \{n+1, \dots, n_0\}\}$. For each vertex E_i , we also denote by $N_i = \{j \in \{1, \dots, N\}: E_i \in k_j\}$ the set of edges adjacent to E_i . The incidence matrix $D = (d_{ij})_{n_0 \times N}$ is defined by

$$d_{ij} = \begin{cases} 1 & \text{if } \pi_j(l_j) = E_i, \\ -1 & \text{if } \pi_j(0) = E_i, \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency matrix $\mathcal{E} = (e_{ih})_{n_0 \times n_0}$ of Γ is given by

$$e_{ih} = \begin{cases} 1 & \text{if there exists an edge } k_{s(i,h)} \text{ between } E_i \text{ and } E_h, \\ 0 & \text{otherwise.} \end{cases}$$

For a function $u: G \rightarrow \mathbb{R}$ we set $u_j = u \circ \pi_j: [0, l_j] \rightarrow \mathbb{R}$ its restriction to the edge k_j . We further use the abbreviations:

$$\begin{cases} u_j(E_i) = u_j(\pi_j^{-1}(E_i)), \\ u_{jx_j}(E_i) = \frac{du_j}{dx_j}(\pi_j^{-1}(E_i)), \\ u_{jx_j^{(n)}}(E_i) = \frac{d^n u_j}{dx_j^n}(\pi_j^{-1}(E_i)). \end{cases}$$

2.2. Data and framework

Following Castro and Zuazua [16], we study a linear system modeling the vibrations of beams connected by point masses but with N beams (instead of two) and n point masses (instead of one). To this end, let us fix a C^4 -network G such that $E_{\text{ext}} \neq \emptyset$. For each edge k_j (representing a beam of our network of beams), we fix mechanical constants: $m_j > 0$ (the mass density of the beam k_j) and $E_j I_j > 0$ (the flexural rigidity of k_j). We set $a_j = \frac{E_j I_j}{m_j}$. For each interior vertex $E_i \in E_{\text{int}}$, we fix the mass $M_i > 0$ ($1 \leq i \leq n$).

So the scalar functions $u_j(x, t)$ and $z_i(t)$ for $x \in G$ and $t > 0$ contain the information on the vertical displacements of the beams ($1 \leq j \leq N$) and of the point masses ($1 \leq i \leq n$). Our aim is to study the spectrum of the spatial operator (involved in the evolution problem) which is defined as follows.

First define the inner product $(\cdot, \cdot)_H$ on $H := \prod_{j=1}^N L^2((0, l_j)) \times \mathbb{R}^n$ by

$$((u, z), (w, s))_H = \sum_{j=1}^N \int_0^{l_j} u_j(x_j) w_j(x_j) dx_j + \sum_{i=1}^n M_i z_i s_i. \quad (1)$$

And define the operator \mathcal{A} on the Hilbert space H endowed with the above inner product, by

$$\begin{cases} D(\mathcal{A}) = \{(u, z) \in H: u_j \in H^4((0, l_j)) \text{ satisfying (2) to (7) hereafter}\}, \\ \forall (u, z) \in D(\mathcal{A}), \quad \mathcal{A}(u, z) = \left((a_j u_{jx_j^{(4)}})_{j=1}^N, -\frac{1}{M_i} \left(\sum_{j \in N_i} a_j \frac{\partial^3 u_j}{\partial v_j^3}(E_i) \right)_{i=1}^n \right) \end{cases} \quad (2)$$

where $\frac{\partial u_j}{\partial v_j}(E_i) = d_{ij} u_{jx_j}(E_i)$ means the exterior normal derivative of u_j at E_i .

$$u_j(E_i) = z_i, \quad \forall i \in I_{\text{int}}, \quad \forall j \in N_i, \quad (3)$$

$$\sum_{j \in N_i} \frac{\partial u_j}{\partial v_j}(E_i) = 0, \quad \forall i \in I_{\text{int}}, \quad (4)$$

$$a_l \frac{\partial^2 u_l}{\partial v_l^2}(E_i) = a_j \frac{\partial^2 u_j}{\partial v_j^2}(E_i), \quad \forall i \in I_{\text{int}}, \quad \forall (l, j) \in N_i^2, \quad (5)$$

$$u_j(E_i) = 0, \quad \forall i \in I_{\text{ext}}, \forall j \in N_i, \quad (6)$$

$$\frac{\partial^2 u_j}{\partial v_j^2}(E_i) = 0, \quad \forall i \in I_{\text{ext}}, \forall j \in N_i. \quad (7)$$

Notice that the conditions (3) imply the continuity of u on G . The conditions (4) and (5) are transmission conditions at the interior nodes and (6) and (7) are boundary conditions.

Lemma 1 (Properties of the operator \mathcal{A}). *The operator \mathcal{A} defined by (2) is a non-negative self-adjoint operator with a compact resolvent.*

Proof. The reason for \mathcal{A} to be a self-adjoint operator with a compact resolvent, is that it is the Friedrichs extension of the triple (H, V, a) defined by

$$V = \left\{ U = (u, z) \in \prod_{j=1}^N H^2((0, l_j)) \times \mathbb{R}^n : \text{satisfying (3), (4), (6)} \right\}$$

which is a Hilbert space endowed with the inner product

$$(U, W)_V = ((u, z), (w, s))_V = \sum_{j=1}^N (u_j, w_j)_{H^2((0, l_j))} + \sum_{i=1}^n M_i z_i s_i$$

where $(\cdot, \cdot)_{H^2((0, l_j))}$ is the usual inner product on $(0, l_j)$ and

$$a(U, W) = \sum_{j=1}^N a_j \int_0^{l_j} u_{jx_j^{(2)}}(x_j) w_{jx_j^{(2)}}(x_j) dx_j. \quad (8)$$

See [32] for the details of the proof. \square

3. Solutions of the wave problem via Fourier series

The aim is to study the controllability of the evolution problem

$$(P) \quad \begin{cases} (u, z)_{tt} + \mathcal{A}(u(t), z(t)) = 0, & t > 0, \\ \text{with } (u(t), z(t)) \in D(\mathcal{A}), \forall t > 0 \text{ and } (\mathcal{A}, D(\mathcal{A})) \text{ defined in Section 2.} \end{cases}$$

Following Castro and Zuazua [16], we will characterize some fractional powers of the linear operator \mathcal{A} , which then allows us to give a description of the solutions of (P) in terms of Fourier series.

In our evolution problem, there are two unknowns:

- $u = (u_j)$ in $\prod_{j=1}^N L^2((0, l_j))$ describes the displacements of the N beams.
- z in \mathbb{R}^n describes the displacements of the n masses located at the interior nodes E_i , $i \in I_{\text{int}}$, i.e. $z_i = u_j(E_i)$, $\forall i \in I_{\text{int}}, \forall j \in N_i$.

Now the vectors of

$$\mathcal{V} := \left\{ u \in \prod_{j=1}^N H^2((0, l_j)) \mid u \text{ satisfies (4) and (6)} \right\}$$

can be identified with those of V , defined in the proof of Lemma 1, by means of the map:

$$\mathcal{V} \rightarrow V$$

$$u \mapsto (u, z) \quad \text{with } z_i = u_j(E_i), \forall i \in I_{\text{int}}, \forall j \in N_i.$$

The norm defined on \mathcal{V} by

$$\|u\|_{\mathcal{V}}^2 = \sum_{j=1}^N a_j \int_0^{l_j} |u_j''(x)|^2 dx$$

is classically equivalent to the norm of $\prod_{j=1}^N H^2((0, l_j))$ on \mathcal{V} . (Recall that the classical arguments to prove this property on an interval are an interpolation inequality and Poincaré inequality. The interpolation inequality is written here on each branch and then the sum is computed. As for Poincaré inequality, it still holds for a network, since the role of the continuity condition is played here by (3).)

The eigenvalue problem associated to Problem (P) can be written as: $\lambda^2 \in \sigma(\mathcal{A})$ ($\lambda > 0$) is an eigenvalue of \mathcal{A} with associated eigenvector $U = (u, z) \in D(\mathcal{A})$ if and only if u satisfies the transmission and boundary conditions (3)–(7) of Section 2 and

$$(EP) \quad \begin{cases} a_j u_{jx_j^{(4)}} = \lambda^2 u_j & \text{on } (0, l_j), \forall j \in \{1, \dots, N\}, \\ \sum_{j \in N_i} a_j \frac{\partial^3 u_j}{\partial v_j^3}(E_i) = \lambda^2 M_i z_i, & \forall i \in I_{\text{int}}, \\ u_j \in H^4((0, l_j)), & \forall j \in \{1, \dots, n\}. \end{cases}$$

The variational formulation of this above eigenvalue problem (EP) is, for any $v \in \mathcal{V}$,

$$(VF) \quad \begin{cases} \sum_{j=1}^N a_j \int_0^{l_j} u_j''(x) v_j''(x) dx = \lambda^2 \left(\sum_{j=1}^N a_j \int_0^{l_j} u_j(x) v_j(x) dx + \sum_{i=1}^N \sum_{j \in N_i} M_i u_j(E_i) v_j(E_i) \right), \\ u \in \mathcal{V}. \end{cases}$$

This formulation has been obtained via two parts integrations (see proof of Lemma 1 in [32] for some more details).

Theorem 1 of [16] for $N = 2$ identical beams with an interior mass equal to 1 still holds for a network of $N > 2$ different beams with different interior masses. Indeed

Proposition 2 (Eigenelements and Fourier series). *There exists a sequence of real positive eigenvalues $(\lambda_k^2)_k$ of the operator \mathcal{A} (defined in Section 2.2) such that*

$$0 < (\lambda_1)^2 \leq (\lambda_2)^2 \leq \dots \quad \text{with} \quad \lim_{k \rightarrow +\infty} \lambda_k = +\infty.$$

The associated eigenfunctions $\hat{\Phi}_k := (\Phi_k, \Phi_k(E_1), \dots, \Phi_k(E_n))$ can be chosen to constitute an orthonormal basis of $H = \prod_{j=1}^N L^2((0, l_j)) \times \mathbb{R}^n$ endowed with the inner product given by (1).

Thus the functions Φ_k constitute an orthogonal system in \mathcal{V} and satisfy $\|\Phi_k\|_{\mathcal{V}}^2 = \lambda_k^2$.

Then \mathcal{V} is characterized as

$$\mathcal{V} = \left\{ (u, z) \mid \exists (\alpha_k)_k \text{ such that } u = \sum_{k \in \mathbb{N}} \alpha_k \Phi_k, \quad \|u\|_{\mathcal{V}}^2 = \sum_{k \in \mathbb{N}} |\alpha_k|^2 (\lambda_k)^2 < \infty \right. \\ \left. \text{and } u_j(E_i) = z_i, \quad \forall i \in I_{\text{int}}, \quad \forall j \in N_i \right\}.$$

And the domains of the powers of the linear operator \mathcal{A} are given by

$$X_\alpha := D(\mathcal{A}^\alpha) = \left\{ (u, z) \mid \exists (\alpha_k)_k, \quad (u, z) = \sum_{k \in \mathbb{N}} \alpha_k \hat{\Phi}_k, \quad \|u\|_{X_\alpha}^2 = \sum_{k \in \mathbb{N}} |\alpha_k|^2 (\lambda_k)^{4\alpha} < \infty \right\}$$

for any α in \mathbb{R} .

Proof. Let us first recall the following classical result:

Lemma 3 (Self-adjoint operator with compact resolvent and orthonormal basis of eigenvectors). Let $A : D(A) \rightarrow H$ be a linear operator with H a Hilbert space and $D(A)$ a dense subspace of H . Assume that A is self-adjoint and that, for any μ in the resolvent set of $(-A)$, $(\mu \cdot \text{Id} + A)^{-1}$ is compact. Then there exists a sequence of real positive eigenvalues $(\mu_k)_k$ of the operator A such that

$$0 < \mu_1 \leq \mu_2 \leq \dots \quad \text{with} \quad \lim_{k \rightarrow +\infty} \mu_k = +\infty.$$

The associated eigenfunctions can be chosen to constitute an orthonormal basis of H .

This lemma is applied to the operator \mathcal{A} (defined in Section 2.2) due to Lemma 1 which gives the required properties.

Now, it follows from the variational formulation (VF) of the eigenvalue problem given above

$$\|\Phi_k\|_{\mathcal{V}}^2 = \lambda_k^2 \left(\sum_{j=1}^N a_j \int_0^{l_j} (\Phi_k)_j(x)^2 dx + \sum_{i=1}^N \sum_{j \in N_i} M_i ((\Phi_k)_j(E_i))^2 \right) = \lambda_k^2 \cdot \|\Phi_k\|_H^2 = \lambda_k^2.$$

Moreover, the eigenfunctions Φ_k constitute an orthogonal system in \mathcal{V} due to this same relationship between the norm in \mathcal{V} and that in H for an eigenfunction. \square

Following Castro and Zuazua [16], we will now give other descriptions of some of the spaces X_α which will be useful for controllability later on.

Proposition 4 (Characterization of the domains of the powers of the spatial operator).

1. $X_0 = D(\mathcal{A}^0)$ coincides algebraically and topologically with the space $(\prod_{j=1}^N L^2((0, l_j))) \times \mathbb{R}^n$, where \mathbb{R}^n is endowed with the norm $\|z\|^2 = \sum_{i=1}^n M_i |z_i|^2$. Furthermore

$$\|(u, z)\|_{X_0}^2 = \left(\sum_{j=1}^N \int_0^{l_j} |u_j(x)|^2 dx \right) + \sum_{i=1}^n M_i |z_i|^2.$$

2. $X_{1/4} = D(\mathcal{A}^{1/4})$ coincides with the subspace of the elements (u, z) of $\mathcal{W} \times \mathbb{R}^n$ satisfying $u_j(E_i) = z_i$, $\forall i \in I_{\text{int}}$, $\forall j \in N_i$, where

$$\mathcal{W} := \left\{ u \in \prod_{j=1}^N H^1((0, l_j)) \mid u \text{ satisfies (6) and } \exists \beta_i, u_j(E_i) = \beta_i, \forall i \in I_{\text{int}}, \forall j \in N_i \right\}.$$

Furthermore

$$\|(u, z)\|_{X_{1/4}}^2 = \left(\sum_{j=1}^N \int_0^{l_j} (|u_j(x)|^2 + |u'_j(x)|^2) dx \right) + \sum_{i=1}^n M_i |z_i|^2.$$

3. $X_{-1/4} = D(\mathcal{A}^{-1/4})$ coincides with the dual space of $X_{1/4}$ i.e. it is the quotient subspace of $\mathcal{W}' \times \mathbb{R}^n$ constituted by the classes (φ, η) characterized in the following way: two elements (φ^1, η^1) and (φ^2, η^2) belong to the same class if and only if

$$(\varphi^1 - \varphi^2, \eta^1 - \eta^2) = \sum_{i \in I_{\text{int}}} \alpha_i \left(\gamma_i, 0, \dots, -\frac{1}{M_i}, \dots, 0 \right)$$

where $\alpha_i \in \mathbb{R}$ and $\langle u, \gamma_i \rangle_{\mathcal{W} \times \mathcal{W}'} = \gamma_i(u) := \max_{j \in N_i} \{u_j(E_i)\}$ for any $i \in I_{\text{int}}$.

Proof. 1. The proof is completely analogous to that of [16]. Also note that $X_0 = D(\mathcal{A}^0) = H$, which is not surprising.

2. The proof is analogous to that of [16] replacing $f : H_0^1 \times \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(u, z) = u(0) - z$ by $f : \mathcal{W} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, defined by $f(u_1, \dots, u_N, z_1, \dots, z_n) = (\alpha_1 - z_1, \dots, \alpha_1 - z_n)$ and the function $x \mapsto 1 - x^2$ in $((1 - x^2), 1)$

by a polynomial function with degree 2 on each branch, which vanishes at the exterior nodes and takes the value 1 at each interior node. Such a function exists since the value at two points of each branch are fixed.

3. The proof is analogous to that of [16] replacing the Dirac distribution at zero δ_0 by γ such that $\sum_{i \in I_{\text{int}}} \alpha_i \gamma_i(u) = \sum_{i \in I_{\text{int}}} \alpha_i (\max_{j \in N_i} \{u_j(E_i)\})$. Thus, for any $(u, z) \in X_{1/4}$, $\sum_{i \in I_{\text{int}}} \alpha_i \gamma_i(u) = \sum_{i \in I_{\text{int}}} \alpha_i z_i$. And this property plays the role of $\delta_0(u) = z$ in [16]. \square

Let us now recall the following Lemma 1.1.6 of [4] which gives the existence, the uniqueness and the regularity of the spectral theoretic solution to any abstract wave equation in a Hilbert space based on a self-adjoint operator. It is applied here to our evolution problem denoted by (EP).

Lemma 5 (Solution in terms of the powers of the operator \mathcal{A}).

1. (Existence, uniqueness, regularity) Assume that $(\mathcal{A}, D(\mathcal{A}))$ is the operator defined in Section 2, $X_\alpha := D(\mathcal{A}^\alpha)$ is the space defined in Proposition 2, for α in \mathbb{R} and $\mathcal{H}_\alpha := X_\alpha \times X_{\alpha-1/2}$ (called energy space).

To every $k \in \mathbb{N}$ and $(U^0, U^1) \in \mathcal{H}_{(k+1)/2}$, exists a solution

$$(u, z) \in C^{k+1-j}([0, +\infty), X_{j/2}), \quad j = 0, \dots, k+1,$$

of the Cauchy problem

$$(CP) \quad \begin{cases} u_{tt} + \mathcal{A}(u(t), z(t)) = 0, & t > 0, \\ (u(0), z(0)) = U^0 = (u^0, z^0), & (u_t(0), z_t(0)) = U^1 = (u^1, z^1). \end{cases}$$

It is unique in $C^2([0, +\infty), X_0) \cap C^1([0, +\infty), X_{1/2}) \cap C^0([0, +\infty), X_1)$ and is given by

$$(u(t), z(t)) = \cos(\mathcal{A}^{1/2}t)(u^0, z^0) + \mathcal{A}^{-1/2} \sin(\mathcal{A}^{1/2}t)(u^1, z^1).$$

2. (Conservation of energy) The solution (u, z) given above, also satisfies, for $j = 0, \dots, k$,

$$E_{(j+1)/2}(t) = \|(u(t), z(t))\|_{X_{(j+1)/2}}^2 + \|(\dot{u}(t), \dot{z}(t))\|_{X_{j/2}}^2 = \text{Const}, \quad \forall t \geq 0.$$

This lemma was generalized by Castro and Zuazua (cf. [16]) to the case $\alpha \in \mathbb{R}$ for two Euler–Bernoulli identical beams connected by a point-mass. It still holds for a network of $N > 2$ different Euler–Bernoulli beams with n interior point-masses.

Proposition 6 (Existence, uniqueness, conservation of energy of the solution). Keep the same assumptions as in Lemma 5. Let $T > 0$.

To every $\alpha \in \mathbb{R}$ and $(U^0, U^1) \in \mathcal{H}_\alpha$, exists a unique solution to the Cauchy problem (CP) (given in Lemma 5), such that $U \in C^0([0, T], \mathcal{H}_\alpha)$.

It is given by

$$U(t) := ((u(t), z(t)), (u_t(t), z_t(t))) = \sum_{k \in Z^*} \alpha_k e^{i\lambda_k t} \bar{\Phi}_k$$

with (λ_k, Φ_k) defined in Proposition 2 and $\alpha_k = \langle ((u^0, z^0), (u^1, z^1)), \bar{\Phi}_k \rangle_{X_{1/2} \times X_0}$ where

$$\bar{\Phi}_k = (\Phi_k, i\lambda_k \Phi_k), \quad k \in Z^*, \quad \text{and} \quad \Phi_{-k} = \Phi_k, \quad \lambda_{-k} = -\lambda_k.$$

Furthermore the energy of the system defined by

$$E_\alpha(t) = \|(u(t), z(t))\|_{X_\alpha}^2 + \|(\dot{u}(t), \dot{z}(t))\|_{X_{\alpha-1/2}}^2$$

is conserved along the time.

Proof. The proof is exactly the same one as in [16] since it is based on the development of the solution in terms of Fourier series which does not change for a higher number of beams. \square

4. General results about controllability applied to a network of Euler–Bernoulli beams

4.1. Controllability and observability

Let us first recall the definition of controllability applied to the problem we will consider. Then we classically establish a sufficient condition called observability inequality.

Let i_0 be an element of I_{ext} and (PC) be the following problem:

$$\left\{ \begin{array}{l} u_{j,tt}(x, t) + a_j u_{jx_j^{(4)}}(x, t) = 0, \quad \forall j \in \{1, \dots, N\}, \\ u_{j,tt}(E_i, t) - \frac{1}{M_i} \left(\sum_{j \in N_i} a_j \frac{\partial^3 u_j}{\partial v_j^3}(E_i) \right) = 0, \quad \forall j \in \{1, \dots, N\}, \forall i \in I_{\text{int}}, \\ u_j(E_i) = z_i, \quad \forall i \in I_{\text{int}}, \forall j \in N_i, \\ \sum_{j \in N_i} \frac{\partial u_j}{\partial v_j}(E_i) = 0, \quad \forall i \in I_{\text{int}}, \\ a_l \frac{\partial^2 u_l}{\partial v_l^2}(E_i) = a_j \frac{\partial^2 u_j}{\partial v_j^2}(E_i), \quad \forall i \in I_{\text{int}}, \forall (l, j) \in N_i^2, \\ u_j(E_i) = 0, \quad \forall i \in I_{\text{ext}}, \forall j \in N_i, \\ \frac{\partial^2 u_j}{\partial v_j^2}(E_i) = 0, \quad \forall i \in I_{\text{ext}} - \{i_0\}, \forall j \in N_i, \quad \text{and} \quad \frac{\partial^2 u_j}{\partial v_j^2}(E_{i_0}) = q, \quad \forall j \in N_{i_0}. \end{array} \right.$$

Note that the control function $q = q(t)$ acts on the system through the exterior node E_{i_0} on the quantity $\frac{\partial^2 u_j}{\partial v_j^2}$.

Definition 7 (Controllability). Problem (PC) is controllable at time $T > 0$ if there exists q in $L^2(0, T)$ such that

$$\left\{ \begin{array}{l} u_j(x, T) = 0, \quad \forall x \in k_j, j \in \{1, \dots, N\}, \\ u_{j,t}(x, T) = 0, \quad \forall x \in k_j, j \in \{1, \dots, N\}, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} z_i(T) = 0, \quad \forall i \in I_{\text{int}}, \\ z_{i,t}(T) = 0, \quad \forall i \in I_{\text{int}}. \end{array} \right.$$

The aim is then to find a class \mathcal{H} of initial conditions $U^0 = (u^0, z^0)$, $U^1 = (u^1, z^1)$ such that Problem (PC) is controllable (recall that U^0 and U^1 are still used for $(u(0), z(0))$ and $(u_t(0), z_t(0))$ respectively as in Lemma 5).

Using the classical Hilbert Uniqueness Method (HUM) developed in Lions (cf. [28]) leads to the following sufficient condition:

Lemma 8 (Observability inequality and controllability). Let $T > 0$. A sufficient condition for problem (PC) to be controllable at time T is the existence of two strictly positive constants K_1 and K_2 such that, if $(U^0, U^1) \in \mathcal{H}$,

$$K_1 \cdot \|(U^0, U^1)\|_{\mathcal{H}}^2 \leq \int_0^T \left| a_j \frac{\partial u_j}{\partial v_j}(E_{i_0}, t) \right|^2 dt \leq K_2 \cdot \|(U^0, U^1)\|_{\mathcal{H}}^2 \quad (9)$$

with $j \in N_{i_0}$.

Note that N_{i_0} only contains one element since E_{i_0} is an exterior node.

Proof. Let us give only the main ideas of the proof. It is based on Hilbert Uniqueness Method (cf. [28]).

First part: A brief presentation of the Hilbert Uniqueness Method.

Consider

1. Ω a domain of \mathbb{R}^n with sufficiently regular boundary Γ ;
2. an elliptic symmetric operator \mathcal{A} with order $2m$ ($m \in \mathbb{N}$) and regular coefficients independent from the variable t ;
3. a set of boundary operators B_j , $j = 1, \dots, m$,

such that the following homogeneous system is well-posed in suitable Hilbert spaces:

$$\begin{cases} \Phi'' + \mathcal{A}\Phi = 0 & \text{in } \Omega \times (0, T), \\ \Phi(0) = \Phi^0, \quad \Phi'(0) = \Phi^1 & \text{in } \Omega, \\ B_j\Phi = 0 & \text{in } \Gamma \times (0, T) \end{cases}$$

with Φ^0 and Φ^1 in $C^\infty(\overline{\Omega})$ satisfying the compatibility conditions $B_j\Phi^0 = 0$, for every j such that the order of B_j is inferior to m .

Define the control problem as:

$$\begin{cases} \Phi'' + \mathcal{A}\Phi = 0 & \text{in } \Omega \times (0, T), \\ \Phi(0) = \Phi^0, \quad \Phi'(0) = \Phi^1 & \text{in } \Omega, \\ B_j\Phi = v_j & \text{in } \Gamma \times (0, T) \end{cases}$$

and the operators C_j by

$$\int_{\Omega} [(\mathcal{A}\Phi)\Psi - \Phi(\mathcal{A}\Psi)] dx = \sum_{j=1}^m \int_{\Gamma} [(C_j\Phi)(B_j\Psi) - (B_j\Phi)(C_j\Psi)] d\Gamma$$

for all $(\Phi, \Psi) \in (C^\infty(\overline{\Omega}) \cap D(\mathcal{A}))^2$ (the exact definition of the C_j 's can be found in [29]).

Then a sufficient condition for the above control problem to be controllable is that

$$\left(\sum_{j=1}^m \|C_j\Phi\|_{L^2(\Gamma \times (0, T))}^2 \right)^{1/2}$$

defines a norm.

Second part: Application to our situation i.e. determination of the operators C_j .

In that paper, the problem is one-dimensional ($n = 1$), \mathcal{A} is the operator defined by (2) to (7) in Section 2, $m = 1$ and the operator B_1 is the differentiation operator at the exterior node E_{i_0} .

Now to establish the Green formula defining the operator C_1 , it is enough to compute $\int_{\Omega} (\mathcal{A}\Phi)\Psi$ which is done via two successive integrations by parts of

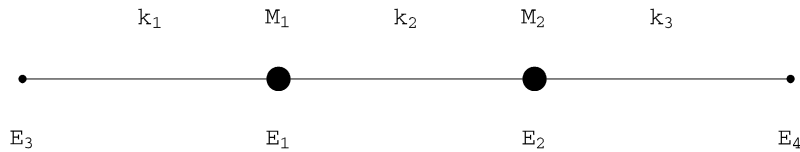
$$\sum_{j=1}^N \int_0^{l_j} a_j u_{jx_j^{(4)}}(x_j) w_j(x_j) dx_j$$

as it has been done in a previous work (cf. [32]). This leads to $C_1 u = a_j \frac{\partial u_j}{\partial v_j}(E_{i_0})$ where j is the only element in N_{i_0} . \square

This first analysis of the problem is a generalization of what Castro and Zuazua do in [16]. The observability inequality (9) was proved there with the space $\mathcal{H} = \mathcal{H}_{1/4}$ in the case of a network with two beams connected by a point mass. It is (5.2) in Proposition 3 of [16]. To prove that inequality, the authors used the properties of the eigenvalues. We will generalize this approach to the case of a chain of N branches in the following sections.

4.2. Observability inequality and spectral gap

Since the solution is expressed in terms of Fourier series (cf. Proposition 6), the observability inequality will be proved using the following result due to Haraux (cf. [24]) and also used by Castro and Zuazua (cf. [16]).

Fig. 1. Graph with $N = 3$ edges.

Lemma 9 (Observability inequality and spectral gap). Let λ_n be a sequence of real numbers such that there exist $(\alpha, \beta, N_0) \in \mathbb{R}^2 \times \mathbb{N}$ satisfying

$$\lambda_{n+1} - \lambda_n \geq \alpha > 0, \quad \forall |n| \geq N_0, \quad (10)$$

and $\lambda_{n+1} - \lambda_n \geq \beta > 0$.

Consider also $T > \pi/\alpha$. Then there exist two constants $C_1(T)$ and $C_2(T)$ which only depend on α , β and N_0 such that, if $f(t) = \sum_{n \in \mathbb{Z}} \alpha_n e^{-i\lambda_n t}$,

$$C_1(T) \sum_{n \in \mathbb{Z}} |\alpha_n|^2 \leq \int_{-T}^T |f(t)|^2 dt \leq C_2(T) \sum_{n \in \mathbb{Z}} |\alpha_n|^2$$

for all $(\alpha_n) \in l^2(\mathbb{R})$.

5. Controllability of a chain of N branches

From now on, we will restrict the study of the controllability to a particular example of graph, that is to say a chain of N branches. We still have interior point masses as described in Section 2. The situation is thus represented by Fig. 1 if $N = 3$.

In general G is the graph with N edges and $(N + 1)$ vertices described by the following adjacency matrix:

$$\mathcal{E} = (e_{ih})_{(N+1) \times (N+1)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & \ddots & \cdots & \vdots & \vdots & 0 \\ 0 & 1 & 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \ddots & \ddots & 0 & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 & 1 & \vdots \\ 0 & \vdots & & & \ddots & 1 & 0 & \vdots \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Now we need to characterize the spectrum $\sigma(\mathcal{A})$ of the operator \mathcal{A} introduced in Section 2 in order to apply the above Lemmas 8 and 9 to get controllability. What is already known from Lemma 1 (see also [16]) is that the spectrum is positive and discrete.

In the following, the first step is to compute explicitly the characteristic function denoted by $f(\sqrt{\lambda})$. Then the asymptotic properties of the spectrum will be studied.

5.1. The characteristic equation

Let us first introduce some useful notation.

Notation. Let u be a non-trivial solution of the eigenvalue problem (EP) (given in Section 3) and λ^2 ($\lambda > 0$) be the corresponding eigenvalue.

For each $j \in \{1, \dots, N\}$, the vector function V_j is defined by

$$V_j(x) = (u_j(x), a_j u_{jx_j^{(2)}}(x), u_{jx_j^{(1)}}(x), a_j u_{jx_j^{(3)}}(x))^t, \quad \forall x \in [0, l_j].$$

Keeping the notation a_j and l_j introduced in Section 2, the matrix A_j is $A_j = A(a_j, b_j)$ with $b_j = a_j^{-\frac{1}{4}} l_j$ and $A(a, b)$ the square matrix of order 4 defined by

$$A(a, b) = \frac{1}{2} \begin{pmatrix} c + ch & \frac{-c+ch}{a^{1/2}\lambda} & \frac{a^{1/4}(s+sh)}{\lambda^{1/2}} & \frac{-s+sh}{a^{1/4}\lambda^{3/2}} \\ a^{1/2}\lambda(-c+ch) & c + ch & a^{3/4}\lambda^{1/2}(-s+sh) & \frac{a^{1/4}(s+sh)}{\lambda^{1/2}} \\ \frac{\lambda^{1/2}(s+sh)}{a^{1/4}} & \frac{s+sh}{a^{3/4}\lambda^{1/2}} & c + ch & \frac{-c+ch}{a^{1/2}\lambda} \\ a^{1/4}\lambda^{3/2}(s+sh) & \frac{\lambda^{1/2}(-s+sh)}{a^{1/4}} & a^{1/2}\lambda(-c+ch) & c + ch \end{pmatrix}$$

with the notation $c = \cos(b\sqrt{\lambda})$, $s = \sin(b\sqrt{\lambda})$, $ch = \cosh(b\sqrt{\lambda})$, $sh = \sinh(b\sqrt{\lambda})$.

The matrix T_j depends on the interior masses M_j (cf. Section 2) and on the eigenvalue λ^2 in the following way:

$$T_j = T(M_j, \lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \lambda^2 M_j & 0 & 0 & 1 \end{pmatrix}.$$

To finish with, the matrix $M(\lambda)$ is given by

$$M(\lambda) = (A_N T_{N-1} A_{N-1}) \dots (A_2 T_1 A_1). \quad (11)$$

Lemma 10 (A few trivial but useful properties). With the notation introduced above:

$$\begin{aligned} V_j(l_j) &= A_j V_j(0), \quad \forall j \in \{1, \dots, N\}, \\ V_{j+1}(0) &= T_j V_j(l_j), \quad \forall j \in \{1, \dots, N-1\}, \\ V_N(l_N) &= M(\lambda) V_1(0). \end{aligned}$$

Proof. Since u_j satisfies the first equation of the eigenvalue problem (EP) (given in Section 3), u_j is a linear combination of the vectors of the fundamental basis

$$(\cos(a_j^{-\frac{1}{4}}\sqrt{\lambda}.), \sin(a_j^{-\frac{1}{4}}\sqrt{\lambda}.), \cosh(a_j^{-\frac{1}{4}}\sqrt{\lambda}.), \sinh(a_j^{-\frac{1}{4}}\sqrt{\lambda})).$$

The first equation of the lemma follows from that property after some calculation.

Now the transmission conditions (3)–(5) of Section 2 and the second equation of (EP) (given in Section 3) imply the second equation.

The third one is the logical consequence of the first two applied successively for $j = 1, j = 2$, etc. \square

Theorem 11 (The characteristic equation for the eigenvalue problem corresponding to a chain of N branches). $\lambda^2 > 0$ is an eigenvalue of \mathcal{A} if and only if λ satisfies the characteristic equation

$$f(\sqrt{\lambda}) = \det(M_{12}(\lambda)) = 0, \quad (12)$$

where $M_{12}(\lambda)$ is the square matrix of order 2 which is the restriction of the matrix $M(\lambda)$, given by (11), to its first two lines and its last two columns.

Proof. Let u be a non-trivial solution of the eigenvalue problem (EP) (given in Section 3) and λ^2 ($\lambda > 0$) be the corresponding eigenvalue. The matrix $M(\lambda)$ is rewritten as

$$M(\lambda) = \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix}$$

where $M_{ij}(\lambda)$ is a square matrix of order 2, for $(i, j) \in \{1, 2\}^2$.

Now, using the boundary conditions (6) and (7) as well as $V_N(l_N) = M(\lambda)V_1(0)$, it follows:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = M_{12}(\lambda) \begin{pmatrix} u_{N_{x_N}^{(1)}}(l_N) \\ a_N u_{N_{x_N}^{(3)}}(l_N) \end{pmatrix}.$$

It is clear that the vector of the second part of the previous equality is non-trivial since u is a non-trivial solution of problem (EP). Hence the result. \square

5.2. Multiplicity of the spectrum

In the problem denoted by (PC) in Section 4.1, the control function q is chosen to act on the system through the exterior node E_{i_0} . In the following, we will assume that the indices of the branches and the nodes of the network are chosen such that E_{i_0} is the only exterior node of the branch k_N (this can be done without loss of generality). Thus the expression $|a_j \frac{\partial u_j}{\partial v_j}(E_{i_0}, t)|$ that we need to estimate to get (9) can also be rewritten as $|a_N u_{N_{x_N}^{(1)}}(l_N)|$. And the first condition which needs to be satisfied is: $u_{N_{x_N}^{(1)}}(l_N) \neq 0$.

The following lemma asserts this result and establishes that the multiplicity of each eigenvalue of \mathcal{A} is one.

Lemma 12. *Let $\mu = \lambda^2 \in \sigma(\mathcal{A})$ (with $\lambda > 0$) be an eigenvalue of the operator \mathcal{A} with associated eigenfunction u . Then the multiplicity of μ is 1 and $u_{N_{x_N}^{(1)}}(l_N) \neq 0$.*

Proof. First notice that all the terms of the matrix $A(a, b)$ defined in Section 5.1 are of the form $c(\lambda, a)(\cosh(b\sqrt{\lambda}) + \epsilon \cos(b\sqrt{\lambda}))$ or $c(\lambda, a)(\sinh(b\sqrt{\lambda}) + \epsilon \sin(b\sqrt{\lambda}))$ where $\epsilon \in \{-1; 1\}$, $c(\lambda, a)$ is strictly positive for any $a > 0$ and $\lambda > 0$, and the functions $\cosh(x) + \epsilon \cos(x)$ and $\sinh(x) + \epsilon \sin(x)$ are strictly positive on $(0; +\infty)$. Thus the terms of the matrix $A(a, b)$ are all strictly positive if a, b and λ are all strictly positive.

The same holds for the matrices $T_j = T(M_j, \lambda)$ for all $j \in \{1, \dots, N\}$.

As a consequence, the matrix $M(\lambda)$ only has strictly positive terms for any $\lambda > 0$. From all that, we deduce that the matrix $M_{12}(\lambda)$ is not the null matrix for any $\lambda > 0$. Then its rank is one and the multiplicity of $\mu = \lambda^2$ is 1.

On the other hand we established in the proof of Theorem 11 that:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = M_{12}(\lambda) \begin{pmatrix} u_{N_{x_N}^{(1)}}(l_N) \\ u_{N_{x_N}^{(3)}}(l_N) \end{pmatrix}.$$

Now if we assume that $u_{N_{x_N}^{(1)}}(l_N) = 0$, then $u_{N_{x_N}^{(3)}}(l_N) \neq 0$ (else u would vanish). But this is equivalent to say that $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of $M_{12}(\lambda)$.

There is a contradiction between that and the fact that all the terms of M_{12} are strictly positive. Hence the result. \square

5.3. Asymptotic behaviour of the spectrum

As announced before, our aim is to use Lemma 9 to get controllability so we need to know the asymptotic behaviour of the spectrum to get Eq. (10) which is a sufficient condition of this lemma. To this end, the asymptotic behaviour of the characteristic function (12) as $\lambda \rightarrow +\infty$ is of great interest.

Proposition 13 (Asymptotic behaviour of the characteristic equation). *Assume that the characteristic equation is still given by Theorem 11. Then there exist three constants $C \neq 0$, C' and C'' which are independent of the variable λ such that:*

$$f(\sqrt{\lambda}) = C \cdot \lambda^{C'} \cdot e^{C''\sqrt{\lambda}} \cdot (f_\infty(\sqrt{\lambda}) + g(\sqrt{\lambda}))$$

where

$$f_\infty(\sqrt{\lambda}) = P(\cos(b_j\sqrt{\lambda}), \sin(b_j\sqrt{\lambda}))_{j \in \{1, \dots, N\}} \quad (13)$$

and P is a polynomial function with $2N$ variables. As for the function g , it satisfies $\lim_{\lambda \rightarrow +\infty} g(\sqrt{\lambda}) = 0$.

Thus, the asymptotic behaviour of the spectrum $\sigma(\mathcal{A})$ corresponds to the roots of the asymptotic characteristic equation

$$f_\infty(\sqrt{\lambda}) = 0. \quad (14)$$

Proof. Using the classical equalities $\cosh(b\sqrt{\lambda}) = \frac{e^{b\sqrt{\lambda}} + e^{-b\sqrt{\lambda}}}{2}$ and $\sinh(b\sqrt{\lambda}) = \frac{e^{b\sqrt{\lambda}} - e^{-b\sqrt{\lambda}}}{2}$ in the computation of the matrix $M(\lambda)$, we see that each term of this matrix is a linear combination of products of some functions which are all of the form: λ^q (where q is a rational number independent of λ), $e^{K\sqrt{\lambda}}$ ($K \in \mathbb{R}$, independent of λ), $\cos(b_j\sqrt{\lambda})$ and $\sin(b_j\sqrt{\lambda})$. Consequently the same property holds for the characteristic function $f(\sqrt{\lambda})$.

Concerning the asymptotic behaviour of the spectrum, since f_∞ is a continuous function on \mathbb{R} (it involves only polynomial, cosine and sine functions) as well as f , then g is also continuous. Thus, if λ_k^2 is the k th eigenvalue of \mathcal{A} , $0 = f(\sqrt{\lambda_k})$ implies

$$f_\infty(\sqrt{\lambda_k}) + g(\sqrt{\lambda_k}) = 0.$$

Now, since $\lim_{k \rightarrow +\infty} \lambda_k = +\infty$ (cf. Proposition 2) and $\lim_{\lambda \rightarrow +\infty} g(\sqrt{\lambda}) = 0$, the asymptotic behaviour of the λ_k^2 's is given by the roots of $f_\infty(\sqrt{\lambda_k}) = 0$. \square

5.4. Spectral gap

As it was said before, our aim is to use Lemma 9 to get controllability so we need to establish that there is a spectral gap for the large eigenvalues of the operator \mathcal{A} (i.e. that Eq. (10) holds) since it is a sufficient condition of this lemma. Since the asymptotic behaviour of the eigenvalues of \mathcal{A} is given by the roots of f_∞ (cf. (14)), it is also sufficient to prove that the roots of f_∞ are simple (thus the distance between two consecutive roots is always superior to a fixed minimal value).

A proof of this assumption denoted by (A) will be given in the case of $N = 2$ branches.

As for the higher values of N , the question remains open. A numerical approach allows one to prove (A) (and get controllability) for many examples with $N = 3$ to $N = 6$.

Case $N = 2$

Using a formal calculation software leads to λ^2 is an eigenvalue of \mathcal{A} if and only if $f(\lambda) = 0$ where

$$f(\sqrt{\lambda}) = \frac{M_1}{\sqrt{\lambda}} \cdot e^{(b_1+b_2)\sqrt{\lambda}} (f_\infty(\sqrt{\lambda}) + r(\lambda))$$

with

$$f_\infty(x) = a_1^{\frac{5}{4}} \sin(b_1 x) \cos(b_2 x) + a_2^{\frac{5}{4}} \cos(b_1 x) \sin(b_2 x) - (a_1^{\frac{5}{4}} + a_2^{\frac{5}{4}}) \sin(b_1 x) \sin(b_2 x)$$

and r is the remainder which satisfies

$$|r(\lambda)| \leq \frac{C}{\sqrt{\lambda}}, \quad \forall \lambda \geq \lambda_0,$$

for some $C > 0$ and $\lambda_0 > 0$. (Recall that all the physical constants M_1, b_1, b_2, a_1 and a_2 are defined in Section 2.2.)

Proposition 14 (Multiplicity of the roots of f_∞). Let f_∞ be the function defined on \mathbb{R} by

$$f_\infty(x) = a_1^{\frac{5}{4}} \sin(b_1 x) \cos(b_2 x) + a_2^{\frac{5}{4}} \cos(b_1 x) \sin(b_2 x) - (a_1^{\frac{5}{4}} + a_2^{\frac{5}{4}}) \sin(b_1 x) \sin(b_2 x)$$

with a_1, a_2, b_1 and b_2 the strictly positive constants given in Section 2.2. Then the roots of f_∞ are simple i.e. there is no real value x_0 satisfying $f_\infty(x_0) = f'_\infty(x_0) = 0$.

Proof. If x_0 is a root of f_∞ with multiplicity at least equal to 2, then $f_\infty(x_0) = f'_\infty(x_0) = 0$ which also reads after some computation

$$\begin{cases} -c_1 + \alpha s_1 + c_2 + s_2 = 0, \\ \alpha \beta c_1 + \beta s_1 + c_2 - s_2 = 0 \end{cases} \quad (15)$$

with

$$\alpha = \frac{a_1^{5/4} - a_2^{5/4}}{a_1^{5/4} + a_2^{5/4}}, \quad \beta = \frac{b_1 - b_2}{b_1 + b_2} \quad (16)$$

and

$$\begin{cases} c_1 = \cos[(b_1 - b_2)x_0], & s_1 = \sin[(b_1 - b_2)x_0], \\ c_2 = \cos[(b_1 + b_2)x_0], & s_2 = \sin[(b_1 + b_2)x_0]. \end{cases} \quad (17)$$

Now, (15) is equivalent to

$$\begin{cases} c_2 = \frac{1}{2}[c_1(1 - \alpha\beta) - s_1(\alpha + \beta)], \\ s_2 = \frac{1}{2}[c_1(1 + \alpha\beta) - s_1(\alpha - \beta)]. \end{cases}$$

Thus the classical equation $c_2^2 + s_2^2 = 1$ can be rewritten as $h(c_1, s_1) = 0$ with h the polynomial function defined on \mathbb{R}^2 by

$$h(x, y) = [x(1 + \alpha\beta) - y(\alpha - \beta)]^2 + [x(1 - \alpha\beta) - y(\alpha + \beta)]^2 - 4. \quad (18)$$

Now, using the Lagrange multiplier method, we will prove that the minimum of the function $(c_1, s_1) \mapsto c_1^2 + s_1^2$ with the constraint $h(c_1, s_1) = 0$ is strictly superior to 1, which contradicts the fact that $c_1^2 + s_1^2 = 1$.

Lemma 15. Let a_1, a_2, b_1 and b_2 be defined as in Section 2.2 and $\alpha, \beta, c_1, c_2, s_1$ and s_2 as in (16) and (17).

The minimum of the function $(c_1, s_1) \mapsto c_1^2 + s_1^2$ subject to the constraint $h(c_1, s_1) = 0$ with h defined by (18) is strictly superior to 1.

Proof. Let the function ϕ be defined on \mathbb{R}^2 by

$$\phi(x, y) = h(x, y) + m(x^2 + y^2)$$

with m the Lagrange multiplier. If (x^*, y^*) is the minimum of $(x, y) \mapsto x^2 + y^2$ subject to the constraint $h(x, y) = 0$, then

$$\begin{cases} \phi'_x(x^*, y^*) = 0, \\ \phi'_y(x^*, y^*) = 0. \end{cases} \quad (19)$$

This system is a linear homogeneous one and since $h(0, 0) \neq 0$, its determinant must vanish. Hence the equation satisfied by the Lagrange multiplier m :

$$4m^2 + 2(1 + \alpha^2)(1 + \beta^2)m + \beta^2(1 + \alpha^2)^2 = 0.$$

There are two real solutions for m : $m_1 = -\frac{\alpha^2+1}{2}$ and $m_2 = -\frac{\beta^2(\alpha^2+1)}{2}$.

First case: $m = m_1$.

Replacing m by its value in one of the equation of (19), it follows from the other equation: $y^* = -\alpha x^*$. Now the constraint $h(x^*, -\alpha x^*) = 0$ leads to $x^* = \pm \frac{\sqrt{2}}{\sqrt{1+2\alpha^2+\alpha^4}}$. Thus the minimum of $(x, y) \mapsto x^2 + y^2$ subject to the constraint $h(x, y) = 0$ is $x^2 + \alpha^2 x^2 = \frac{2}{1+\alpha^2}$.

Since a_1, a_2, b_1 and b_2 are all strictly positive, $\alpha^2 < 1$ and $\beta^2 < 1$.

Then $\frac{2}{1+\alpha^2} > 1$.

Second case: $m = m_2$.

This time $y^* = \frac{1}{\alpha} x^*$ and the constraint leads to $x^* = \pm \frac{\sqrt{2}}{\beta\sqrt{1+2/\alpha^2+\alpha^2}}$. The minimum is $\frac{2}{\beta^2(1+\alpha^2)}$ and since $\alpha^2 < 1$ and $\beta^2 < 1$, it is always strictly superior to 1. \square

To finish with the proof of the proposition, it is now clear that, if $h(x, y) = 0$, it never holds $x^2 + y^2 = 1$. \square

Proposition 16 (Distance between two consecutive roots of f_∞). Let f_∞ be the function defined on \mathbb{R} by

$$f_\infty(x) = a_1^{\frac{5}{4}} \sin(b_1 x) \cos(b_2 x) + a_2^{\frac{5}{4}} \cos(b_1 x) \sin(b_2 x) - (a_1^{\frac{5}{4}} + a_2^{\frac{5}{4}}) \sin(b_1 x) \sin(b_2 x)$$

with a_1, a_2, b_1 and b_2 the strictly positive constants given in Section 2.2.

There exists $h_0 > 0$ such that, for any $(x, y) \in \mathbb{R}^2$ with $y > x$ and $f_\infty(x) = f_\infty(y) = 0$,

$$|y - x| \geq h_0 > 0.$$

Proof. We established in the proof of Proposition 14 that the system (15) subject to the constraints $c_1^2 + s_1^2 = 1$ and $c_2^2 + s_2^2 = 1$ has no solution. Thus, defining the set S by

$$S = \{(c_1, s_1, c_2, s_2) \in \mathbb{R}^4 : (c_1, s_1, c_2, s_2) \text{ satisfying the following system (20)}\},$$

$$\begin{cases} -c_1 + \alpha s_1 + c_2 + s_2 = 0, \\ c_1^2 + s_1^2 = 1, \\ c_2^2 + s_2^2 = 1, \end{cases} \quad (20)$$

it holds

$$\min_{(c_1, s_1, c_2, s_2) \in S} (\alpha \beta c_1 + \beta s_1 + c_2 - s_2)^2 = d^2 > 0.$$

Which means nothing else than: any x_0 such that $f_\infty(x_0) = 0$ satisfies $|f'_\infty(x_0)| \geq d > 0$.

On the other hand, since f_∞ and all its derivatives are trigonometric polynomials, they are all bounded on \mathbb{R} . Then

$$\forall x \in \mathbb{R}, \quad |f'_\infty(x+h) - f'_\infty(x)| = |f''_\infty(x+\theta h)| \cdot |h| \leq \|f''_\infty\|_\infty \cdot |h|$$

and it follows that f'_∞ is uniformly continuous on \mathbb{R} .

Thus, there exists $h_0 > 0$ such that, for any x_0 satisfying $f_\infty(x_0) = 0$,

$$|x - x_0| \leq h_0 \Rightarrow |f'_\infty(x)| \geq \frac{d}{2}.$$

Due to Rolle's Theorem, this property implies that x_0 is the unique root of f_∞ in the interval $[x_0 - h_0, x_0 + h_0]$, which also means that the minimal distance between two consecutive roots of f_∞ is h_0 . \square

Case $N = 3$

Using a formal calculation software leads to λ^2 is an eigenvalue of \mathcal{A} if and only if $f(\lambda) = 0$ where

$$f(\lambda) = \frac{M_1 M_2}{32(a_1 a_3)^{(7/4)} a_2^3} \cdot e^{(b_1+b_2+b_3)\sqrt{\lambda}} (f_\infty(\sqrt{\lambda}) + r(\lambda))$$

with

$$\begin{aligned} f_\infty(x) = & a_1^{(5/4)} a_2^{(5/4)} \sin(b_1 x) \cos(b_2 x) \cos(b_3 x) + a_2^{(5/4)} a_3^{(5/4)} \cos(b_1 x) \cos(b_2 x) \sin(b_3 x) \\ & - (a_1^{(5/4)} a_2^{(5/4)} + a_2^{(5/2)}) \sin(b_1 x) \sin(b_2 x) \cos(b_3 x) \\ & - (a_1^{(5/4)} a_2^{(5/4)} + a_2^{(5/2)}) \sin(b_1 x) \sin(b_2 x) \cos(b_3 x) \\ & - (a_2^{(5/4)} a_3^{(5/4)} + a_2^{(5/2)}) \cos(b_1 x) \sin(b_2 x) \sin(b_3 x) + a_2^{(5/2)} \cos(b_1 x) \sin(b_2 x) \cos(b_3 x) \\ & + (a_1^{(5/4)} a_2^{(5/4)} + a_2^{(5/2)} + a_1^{(5/4)} a_2^{(5/4)}) \sin(b_1 x) \sin(b_2 x) \sin(b_3 x) \end{aligned}$$

and r is the remainder which satisfies

$$|r(\lambda)| \leq \frac{C}{\sqrt{\lambda}}, \quad \forall \lambda \geq \lambda_0,$$

for some $C > 0$ and $\lambda_0 > 0$. (Recall that all the physical constants M_1, b_1, b_2, a_1 and a_2 are defined in Section 2.2.)

Proving that the zeroes of f_∞ are simple is not an easy task. In fact the general question remains open. Nevertheless a beginning of an answer is given by the numerical study of some examples. First note that we can consider two cases.

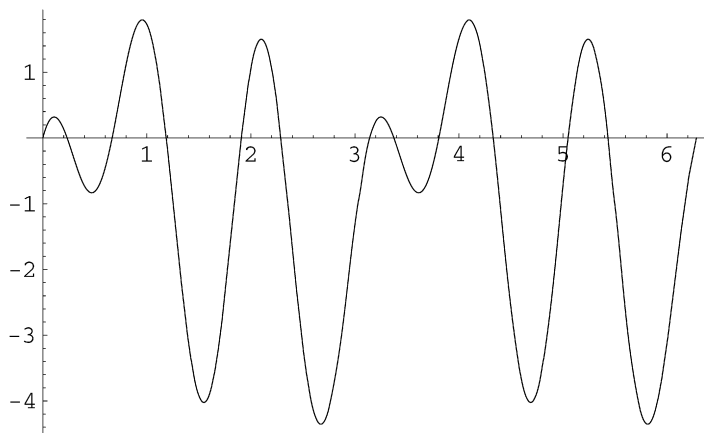


Fig. 2. $N = 3$, $a_i = 1$, $b_i = i$, $i = 1, \dots, 3$.

First case: The ratios (b_j/b_3) for $j = 1$ and $j = 2$ are rational numbers

In this case the function f_∞ is periodic and a direct analysis or a simple representation of its graph shows that the roots are simple.

Example 1. Consider $a_j = b_j = 1$ for any $j \in \{1, 2, 3\}$. Then

$$f_\infty(x) = \sin(x)(3 - 4 \sin(2x)).$$

Denote by $\omega_0 = 0$, $\omega_1 = \frac{\arcsin(3/4)}{2}$ and $\omega_2 = \frac{(\pi/2) - \arcsin(3/4)}{2}$ the angles in $[0, \pi[$ such that $f_\infty(\omega_l) = 0$ for $l = 0, 1, 2$, enumerated in increasing order. Proposition 13 implies that the spectrum $\{\mu_k\}_{k \in \mathbb{N}^*}$ satisfies

$$\mu_{3k+l} - (\omega_l + k\pi)^4 \rightarrow 0, \quad \text{as } k \rightarrow +\infty, \quad \forall l \in \{1, \dots, 3\}.$$

Example 2. Consider $a_j = 1$ and $b_j = j$, for $j \in \{1, 2, 3\}$ (see Fig. 2). Then

$$f_\infty(x) = 2 \sin(x)(2 \cos(x) - \sin(x) + \cos(3x) - \sin(3x) - 2 \sin(5x)).$$

In this example, the period is (2π) . The representation of f_∞ shows that the roots are simple.

Note that there are 12 roots of f_∞ in the interval $[0, 2\pi[$. Denoting by ω_l , with $l \in \{0, \dots, 11\}$, these roots enumerated in increasing order, we deduce that the spectrum $\{\mu_k\}_{k \in \mathbb{N}^*}$ satisfies

$$\mu_{12k+l} - (\omega_l + 2k\pi)^4 \rightarrow 0, \quad \text{as } k \rightarrow +\infty, \quad \forall l \in \{0, \dots, 11\}.$$

Second case: At least one of the ratios (b_j/b_3) for $j = 1$ and $j = 2$ is not a rational number

This time the function f_∞ is not periodic. In order to check numerically that its roots are simple, we proceed as follows:

Let us consider the function g defined on \mathbb{R} by $g(x) = f_\infty(x)^2 + (\frac{df_\infty}{dx}(x))^2$.

Replacing the angles $(b_j x)$ in g by θ_j for $j \in \{1, 2, 3\}$, we get a function G of 3 variables which satisfies

$$G(b_1 x, b_2 x, b_3 x) = g(x).$$

More precisely, for all $(\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$, $G(\theta_1, \theta_2, \theta_3) \in \mathbb{R}^+$ and is (2π) -periodic with respect to each variable. Let us denote by $d \in \mathbb{R}^+$ the minimum of G i.e.

$$d = \min_{(\theta_1, \theta_2, \theta_3) \in [0, 2\pi]^3} G(\theta_1, \theta_2, \theta_3).$$

If $d > 0$, then $g(x) \neq 0$, $\forall x \in \mathbb{R}$, and the roots of f_∞ are consequently simple.

Now the numerical computation of d is done with several examples and given in Table 1. Note that d is always strictly positive.

Table 1

a_1	a_2	a_3	b_1	b_2	b_3	d
1	1	1	1	1	$\sqrt{2}$	0.079
1	$\sqrt{5}$	3	1	1	$\sqrt{2}$	2.2449

Some examples for the cases $N \geq 4$

Due to the complexity of the characteristic equation it is not possible to compute and analyze f_∞ in general. In spite of everything with the help of a formal calculation software we get the expression of f_∞ for $N = 4, 5$ and 6 and for identical beams (i.e. $a_j = b_j = 1$, for $j \in \{1, \dots, N\}$). Here they are:

For $N = 4$,

$$f_\infty(x) = \sin(x)(\cos(x) - \sin(x))(1 - 2\sin(2x)).$$

For $N = 5$,

$$f_\infty(x) = \sin(x)(5 - 20\sin(2x) + 16\sin^2(2x)).$$

For $N = 6$,

$$f_\infty(x) = \sin(x)(\cos(x) - \sin(x))(1 - 4\sin(2x))(3 - 4\sin(2x)).$$

Note that in all these cases a simple analysis shows that the roots of f_∞ are all simple.

Conjecture for a generalization

What we know in general, that is to say for any value of N , is that, when the ratios (b_j/b_N) , for $j \in \{1, \dots, N-1\}$ are rational numbers, f_∞ is always periodic.

Moreover the study of f in the particular cases $N = 2$ and $N = 3$ allows to conjecture that f is of the form

$$f(\lambda) = c \left(\prod_{i=1}^N M_i(\sqrt{\lambda}) \right)^{\frac{N-3}{2}} (f_\infty(\sqrt{\lambda}) + r(\lambda)) \quad \text{with } c \neq 0.$$

Of course this conjecture is not easy to prove. Here are some examples for which we have computed the expression $\frac{1}{(\prod_{i=1}^N M_i)x^{N-3}} f(x^2)$ on an interval of the form $[x_0, x_0 + T]$ where T is the period of f_∞ and x_0 large enough such that the representation that we obtain corresponds to the graph of f_∞ .

Example 3. Consider $a_j = 1$, $b_j = j$ for $j \in \{1, \dots, 4\}$ and $[x_0, x_0 + T] = [100\pi, 102\pi]$ (see Fig. 3).

Example 4. Consider $a_1 = a_4 = 1$, $a_2 = a_3 = \frac{1}{2}$, $b_1 = b_4 = 2$, $b_2 = b_3 = \frac{1}{2}$ and $[x_0, x_0 + T] = [100\pi, 102\pi]$ (see Fig. 4).

5.5. Controllability for a chain of N different branches

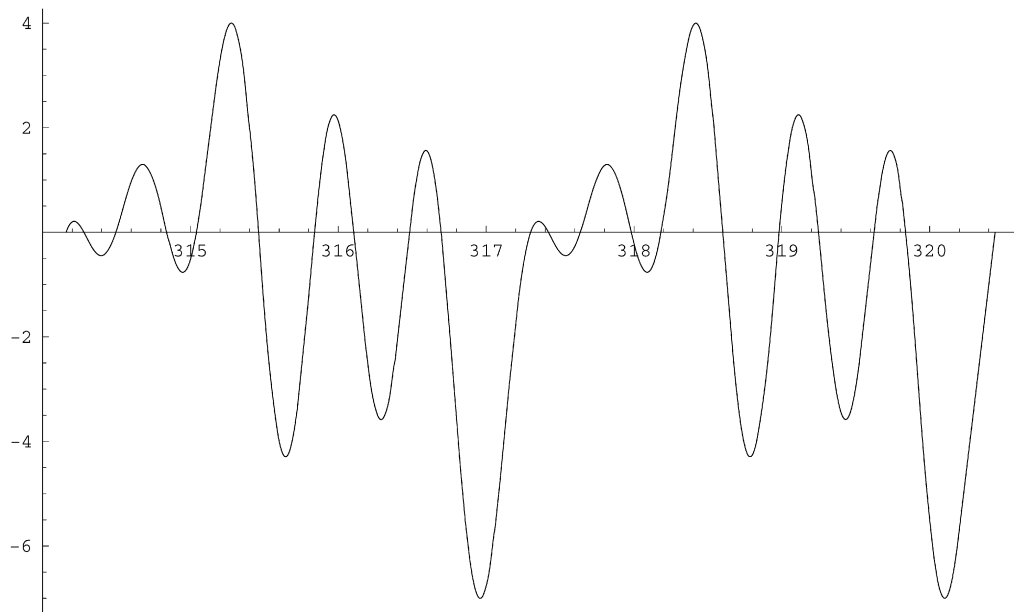
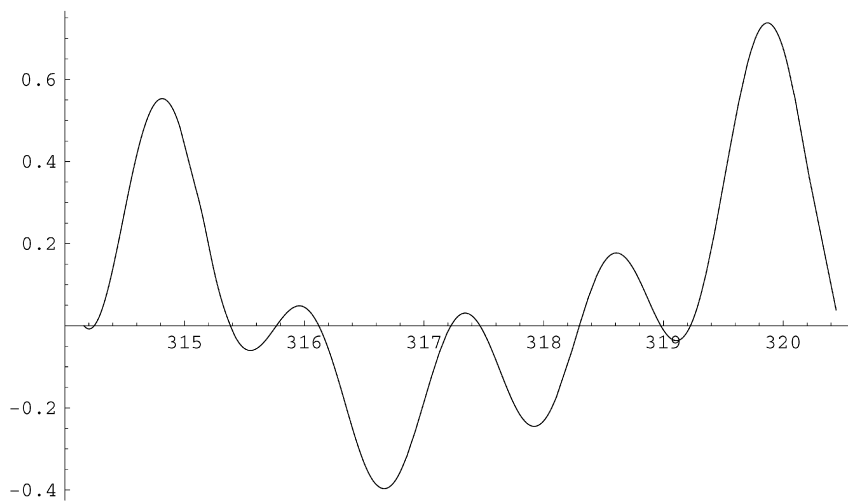
All the required elements to prove controllability in the case of a network with N different branches are now available. The following theorem is a generalization of Theorem 7 of [16].

Theorem 17 (Controllability). Let T be strictly positive and consider the initial data (U^0, U^1) in $\mathcal{H}_{1/4}$ (cf. Lemma 5 and Proposition 6).

Assume that the roots of the function f_∞ are all simple where f_∞ is defined by (13) in Proposition 13 (it describes the asymptotic behaviour of the spectrum of the operator \mathcal{A} given in Section 2).

Then there exists a control $q(t)$ in $L^2(0, T)$ such that the solution of Problem (PC) given in Section 4.1 satisfies

$$\begin{cases} u_j(x, T) = 0, & \forall x \in k_j, j \in \{1, \dots, N\}, \\ u_{j,t}(x, T) = 0, & \forall x \in k_j, j \in \{1, \dots, N\}, \end{cases}$$

Fig. 3. $N = 4$, $a_j = 1$, $b_j = j$, $j = 1, \dots, N$.Fig. 4. $N = 4$, $a_1 = a_4 = 1$, $a_2 = a_3 = \frac{1}{2}$, $b_1 = b_4 = 2$, $b_2 = b_3 = \frac{1}{2}$.

and

$$\begin{cases} z_i(T) = 0, & \forall i \in I_{\text{int}}, \\ z_{i,t}(T) = 0, & \forall i \in I_{\text{int}}. \end{cases}$$

Proof. Following Castro and Zuazua i.e. applying the Hilbert Uniqueness Method (recalled in Section 4.1), the control problem is reduced to the obtention of the observability inequality (9) for the uncontrolled problem that is to say for Problem (PC) with $q = 0$. Using the representation of the solution as a Fourier series, it is equivalent to show the existence of the spectral gap defined in Lemma 9. Now Proposition 16 gives exactly the required gap for the eigenvalues and it is clear that the assumption of simplicity for the roots of f_∞ is sufficient for this proposition to hold (not only for $N = 2$). \square

Corollary 18 (*List of the cases when controllability holds*). *Keeping the notations of Section 2.2, the assumptions of above Theorem 17 are satisfied and so, controllability holds in the sense recalled in the theorem for all the following cases:*

1. $N = 2$.
2. $N \geq 3$ and the ratios (b_j/b_N) , for $j \in \{1, \dots, N-1\}$ are rational numbers.
3. $N = 3$ and $a_1 = a_2 = a_3 = b_1 = b_2 = 1$, $b_3 = \sqrt{2}$.
4. $N = 3$ and $a_1 = b_1 = b_2 = 1$, $a_2 = \sqrt{5}$, $a_3 = 3$, $b_3 = \sqrt{2}$.
5. $N = 4$ and $a_1 = a_2 = a_3 = a_4 = 1$, $b_1 = 1$, $b_2 = 2$, $b_3 = 3$, $b_4 = 4$.
6. $N = 4$ and $a_1 = a_4 = 1$, $a_2 = a_3 = \frac{1}{2}$, $b_1 = b_4 = 2$, $b_2 = b_3 = \frac{1}{2}$.

Proof. It follows from the analysis of Section 5.4 where the roots of the function f_∞ were proved to be all simple in all these different cases. The above theorem is then applied to get the exact controllability. \square

As it was said in Section 5.4, the general problem of controllability remains open for $N \geq 3$ even if no counterexample has been found.

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